Int. J. Solids Structures, 1966, Vol. 2, pp. 337 to 350. Pergamon Press Ltd. Printed in Great Britain

FINITE THERMAL OSCILLATIONS OF THIN PLATES

R. K. KAUL
International Business Machines Corporation, Thomas J. Watson Research Center
Yorktown Heights, New York

Abstract—A system of two-dimensional thermal equations of motion for anisotropic thin plates is obtained from the non-linear three-dimensional oscillation theory of finite elasticity, by using the method based on expansion of the displacement functions in terms of thickness coordinates and the variational method of Kirchhoff. Non-linearity, based on geometric considerations is expressed by taking into account the angles of rotation in the definition of strain tensor, while physical linearity is expressed by using Duhamel's law of anisotropic thermoelasticity generalized to large deformations.

INTRODUCTION

IN THIS paper, equations governing the large flexural vibrations of anisotropic thin plates, including rotatory inertia, shear deformation and temperature distribution are derived from the point of view of the non-linear three-dimensional theory of finite elasticity. Linear equations of equilibrium of an isotropic thin plate subjected to temperature distribution $\theta = \theta^{(0)} + x_2\theta^{(1)}$ have been previously considered by Nádai [1]. Also, isothermal theory of motion of anisotropic thin plates based on infinitesimal theory of deformation has been considered by Voigt, Cauchy and Mindlin [2–4]. In the case of large flexural displacements, the basic theory available in literature is due to Föppl and Kármán [5, 6], who have considered only the isothermal equation of equilibrium of isotropic, thin, elastic plates. The present paper, therefore, extends the work of Föppl and Kármán to the case of anisotropic plates, with large amplitude of vibration, and subjected to temperature distribution of the same form as considered by Nádai. In the absence of temperature field, inertia forces, and shear deformation, these equations reduce to the Kármán equations of equilibrium of isotropic plates. If, in addition, the extensional forces in the plane of the plate be considered constants, then these equations degenerate to the equations of equilibrium under edge thrusts [7]. On the other hand, by dropping only the thermal and the non-linear terms, Timoshenko beam or Mindlin plate equations for isotropy or Mindlin plate equations for anisotropy are obtained from these equations [8–10].

The system of two-dimensional equations of motion is obtained from the three-dimensional non-linear theory of elasticity, by using the method based on series expansion of the displacement functions in terms of the thickness coordinates and the variational method of Kirchhoff [11]. No comparison is made with the exact theory, since the frequency spectra of the exact non-linear theory is not known. However, judging from the recent results in the linear theory of vibrations of thin plates [12], solutions of these equations for bounded plates, should give better results over an extended range of frequencies as compared to the results from the existing theories.

THREE-DIMENSIONAL EQUATIONS

Using Cartesian system of coordinates and Lagrangian concept of strain, the strain tensor γ_{ij} in terms of displacements u_i is given by the relation [12, 13]

$$\gamma_{ij} = e_{ij}+\tfrac{1}{2}(e_{ki}+\omega_{ki})(e_{kj}+\omega_{kj}), \tag{1}$$

where

$$e_{ki} = \tfrac{1}{2}(u_{k,i}+u_{i,k}) = e_{ik},$$
$$\omega_{ki} = \tfrac{1}{2}(u_{k,i}-u_{i,k}) = -\omega_{ik}.$$

In these equations, indices following a comma denote partial derivatives with respect to Cartesian coordinates and every Latin index runs from 1 to 3. Greek indices, however take the values 1 and 3 and are considered cyclic when they occur as 1,3 and anticyclic as 3,1. Summation convention is implied everywhere, unless stated to the contrary, or when indices are put within parentheses.

From the expression for strain it is easy to see the conditions under which the finite strain tensor γ_{ij} can be identified with the linear expression e_{ij}, as is usually done in the classical theory of elasticity. One can simplify these relations in two cases: (1) when the displacements are small in comparison with the dimensions of the body and rotations are small in comparison with unity, and (2) when the extensions and shears are both small in comparison with unity. It may be noticed that case (1) is more restrictive since it implies case (2), and is accepted generally in the classical theory of infinitesimal elasticity, where γ_{ij} is identified with e_{ij}. However, case (2) does not imply case (1). When both extensions and shears are small compared to unity, then products such as $e_{ki}\omega_{kj}$, $\omega_{ki}e_{kj}$, or $e_{ki}e_{kj}$ can be neglected in comparison with $\omega_{ki}\omega_{kj}$ and the strain tensor can be simplified to [13]:

$$\gamma_{ij} \sim e_{ij}+\tfrac{1}{2}\omega_{ki}\omega_{kj}, \tag{2}$$

and is considered as the principal part of Lagrangian strain tensor. The expression for strain components may be further simplified in special cases. In the case of thin plates, which can be considered as rigid enough in its own plane, say (x_1, x_3), the rotation ω_{13} may be considered as negligibly small in comparison with the other two components [13].

When elongations and shears are both small compared to unity, the differences in the dimensions of an elementary cube before and after deformation can be ignored. This then allows us to refer the stresses, body forces, acceleration forces and mass density to the initial area and volume of the element. This, therefore, implies that we take into account only the rotation of the cube and ignore its deformation and thus consider that the cube after deformation differs from its pre-deformed state only in its position in space. The equations of equilibrium based on these approximations will, therefore, be valid only for the case of small relative deformation and arbitrary rotations.

When the form of strain-energy function W is known, the equations of motion and boundary conditions may be deduced from Hamilton's principle [14], which states that

$$\delta\int_{t_0}^{t_1}(T-V)\,dt+\int_{t_0}^{t_1}\delta\Omega\,dt = 0. \tag{3}$$

This says that the variation of the integral of the Lagrangian function $(T-V+\Omega)$, between times t_0 and t_1, takes a stationary value, provided the variation of the displacement vector is taken in such a way that it vanishes at the times t_0 and t_1. Here T is the kinetic energy, which is the volume integral of $\frac{1}{2}\rho\dot{u}_i\dot{u}_i$ and V is the potential energy of deformation and hence the volume integral of strain energy W. Further, $\delta\Omega$ is the work done by the body forces X_i and surface forces f_i when the displacement undergoes a variation δu_i. Following Love, it can then be shown that the variational equation takes the form

$$\int_V [\delta W+\rho(\ddot{u}_i-X_i)\delta u_i]\,\mathrm{d}V = \int_S f_i\delta u_i\,\mathrm{d}S \tag{4}$$

where dots represent differentiation with respect to time, and

$$\delta W = \tau_{ij}\delta\gamma_{ij}, \qquad \tau_{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial\gamma_{ij}}+\frac{\partial W}{\partial\gamma_{ji}}\right).$$

Since the principal part of $\delta\gamma_{ij}$ is not the same as the variation of the principal part of γ_{ij}, it is necessary to retain terms of the type $e_{ki}\omega_{kj}$ in the expression for γ_{ij}. Only then is it possible to later reduce the equations to the case when both elongations and shears are small compared to unity. Thus in the variational equation of motion we use

$$\gamma_{ij} = e_{ij}+\tfrac{1}{2}(e_{ki}\omega_{kj}+e_{kj}\omega_{ki}+\omega_{ki}\omega_{kj}) \tag{5}$$

instead of equation (2). Then substituting (5) in (4), taking the variation with respect to displacement u_i, using divergence theorem, and rearranging the indices suitably through proper use of summation convention, leads to the variational equation

$$\int_V [(\tau_{ij}+\omega_{ik}\tau_{kj}+\tfrac{1}{2}e_{ik}\tau_{kj}-\tfrac{1}{2}e_{jk}\tau_{ki})_{,j}+\rho X_i-\rho\ddot{u}_i]\delta\ddot{u}_i\,\mathrm{d}V$$
$$= \int_S [v_j(\tau_{ij}+\omega_{ik}\tau_{kj}+\tfrac{1}{2}e_{ik}\tau_{kj}-\tfrac{1}{2}e_{jk}\tau_{ki})-f_i]\delta u_i\,\mathrm{d}S. \tag{6}$$

For infinitesimal strains, the quantities $e_{ik}\tau_{kj}$ and $e_{jk}\tau_{ki}$ are of the same order of smallness and can be neglected in comparison with $\omega_{ik}\tau_{kj}$. This then leads to the simpler equations

$$\int_V [(\tau_{ij}+\omega_{ik}\tau_{kj})_{,j}+\rho X_i-\rho\ddot{u}_i]\delta u_i\,\mathrm{d}V = \int_S [v_j(\tau_{ij}+\omega_{ik}\tau_{kj})-f_i]\delta u_i\,\mathrm{d}S. \tag{7}$$

Now the coefficients of the variation δu_i under the integral sign must vanish separately over the surface S and also at all points in the interior of the volume V, since the variation δu_i is quite arbitrary. We thus get the three non-linear equations of motion

$$(\tau_{ij}+\omega_{ik}\tau_{kj})_{,j}+\rho X_i = \rho\ddot{u}_i \tag{8a}$$

and three conditions on the surface

$$(\tau_{ij}+\omega_{ik}\tau_{kj})v_j = f_i \tag{8b}$$

where $v_j = \cos(v, x_j)$; v being the outward drawn normal to the surface S.

PLATE EQUATIONS OF EQUILIBRIUM

We now develop the two-dimensional counterpart of the three-dimensional equations of motion of a thin plate of constant thickness h and mass density ρ, and bounded by a smooth closed contour and parallel plane faces. The middle plane is defined by x_1–x_3 axis and the normal to the face lies parallel to x_2. Since the thickness of the plate is small compared to the other dimensions, we develop the displacement function u_i in powers of thickness coordinate x_2. We thus write

$$u_i = \sum_n x_2^n u_i^{(n)}(x_1, x_3, t), \tag{9a}$$

$$\gamma_{ij} = \sum_n x_2^n \gamma_{ij}^{(n)}(x_1, x_3, t), \tag{9b}$$

$$\omega_{ij} = \sum_n x_2^n \omega_{ij}^{(n)}(x_1, x_3, t), \tag{9c}$$

and use the variational equation (7) to eliminate the dependence of the thickness coordinate x_2, which is now explicitly contained in the integrands.

Substituting the power expansions (9) in the volume integral in (7), carrying out the process of integration with respect to x_2, and then setting in the area integral, the coefficients of the variation $\delta u_i^{(n)}$ equal to zero, we obtain the n-triplets of differential equations:

$$[T_{i\alpha}^{(n)} + \sum_m \omega_{ik}^{(m)} T_{k\alpha}^{(m+n)}]_{,\alpha} - n[T_{i2}^{(n-1)} + \sum_m \omega_{ik}^{(m)} T_{k2}^{(m+n-1)}] + [G_{i2}^{(n)} + \sum_m \omega_{ik}^{(m)} G_{k2}^{(m+n)}] + \rho X_i^{(n)} = \rho \sum_m B_{mn} \ddot{u}_i^{(m)} \tag{10}$$

where

$$T_{i\alpha}^{(n)} = \int_{-h/2}^{h/2} x_2^n \tau_{i\alpha}\, dx_2,$$

$$G_{i2}^{(n)} = [x_2^n \tau_{i2}]_{-h/2}^{h/2} = (h/2)^n[\tau_{i2}|_{+h/2} + (-1)^{n+1}\tau_{i2}|_{-h/2}],$$

$$X_i^{(n)} = \int_{-h/2}^{h/2} x_2^n X_i\, dx_2,$$

$$B_{mn} = \frac{(h/2)^{m+n+1}}{m+n+1}[1+\cos(m+n)\pi];\ m, n = 0, 1, 2, \ldots.$$

Similarly from the surface integral in (7), after substituting the power expansions (9), conducting the process of integration with respect to x_2, and then setting in the resulting contour integral the coefficients of the variation $\delta u_i^{(n)}$ equal to zero, we obtain

$$f_i(A) = G_{i2}^{(n)} + \sum_m \omega_{ik}^{(m)} G_{k2}^{(m+n)} \tag{11a}$$

on the plane parallel faces of the plate, and

$$f_i(s) = \nu_\alpha[T_{i\alpha}^{(n)} + \sum_m \omega_{ik}^{(m)} T_{k\alpha}^{(m+n)}] \tag{11b}$$

on the closed contour bounding the plate. Here $\nu_\alpha = \varepsilon_{\alpha\beta}\, \partial x_\beta/\partial s$ where $\varepsilon_{\alpha\beta}$ is the alternating surface tensor.

The two-dimensional stress equations of motion and the boundary conditions of various orders are then written as:

(a) Zero-order equations, ($n = 0$)

$$[T_{i\alpha}^{(0)} + \sum_m \omega_{ik}^{(m)} T_{k\alpha}^{(m)}]_{,\alpha} + [G_{i2}^{(0)} + \sum_m \omega_{ik}^{(m)} G_{k2}^{(m)}] + \rho X_i^{(0)} = \rho \sum_m B_{m0} \ddot{u}_i^{(m)}, \tag{12a}$$

$$f_i(A) = G_{i2}^{(0)} + \sum_m \omega_{ik}^{(m)} G_{k2}^{(m)}, \tag{12b}$$

$$f_i(s) = \nu_\alpha[T_{i\alpha}^{(0)} + \sum_m \omega_{ik}^{(m)} T_{k\alpha}^{(m)}]. \tag{12c}$$

(b) First-order equations, ($n = 1$)

$$[T_{i\alpha}^{(1)} + \sum_m \omega_{ik}^{(m)} T_{k\alpha}^{(m+1)}]_{,\alpha} - [T_{i2}^{(0)} + \sum_m \omega_{ik}^{(m)} T_{k2}^{(m)}] + [G_{i2}^{(1)} + \sum_m \omega_{ik}^{(m)} G_{k2}^{(m+1)}] + \rho X_i^{(1)} = \rho \sum_m B_{m1} \ddot{u}_i^{(m)}, \tag{13a}$$

$$f_i(A) = G_{i2}^{(1)} + \sum_m \omega_{ik}^{(m)} G_{k2}^{(m+1)}, \tag{13b}$$

$$f_i(s) = \nu_\alpha [T_{i\alpha}^{(1)} + \sum_m \omega_{ik}^{(m)} T_{k\alpha}^{(m+1)}], \tag{13c}$$

and so on for $n = 2, 3, 4, \ldots$.

STRAIN DISPLACEMENT RELATIONS

In order to be able to express the stress equations of motion in terms of displacement components, we need the development of $\gamma_{ij}^{(n)}$ and $\omega_{ij}^{(n)}$ in terms of displacement components $u_i^{(n)}$. Then from equations (2) and (9), we obtain

$$\gamma_{ij} = \sum_n x_2^n \{e_{ij}^{(n)} + \tfrac{1}{2} \sum_m x_2^m \omega_{ki}^{(n)} \omega_{kj}^{(m)}\} = \sum_n x_2^n \gamma_{ij}^{(n)}.$$

The nth-order strain component $\gamma_{ij}^{(n)}$ is then defined as

$$\gamma_{ij}^{(n)} = \{e_{ij}^{(n)} + \tfrac{1}{2} \sum_{m=0}^{n} \omega_{ki}^{(n-m)} \omega_{kj}^{(m)}\}, \tag{14a}$$

where

$$e_{ij}^{(n)} = \tfrac{1}{2}\{(u_{i,\alpha}^{(n)} \delta^{\alpha j} + u_{j,\alpha}^{(n)} \delta^{\alpha i}) + (n+1)(u_i^{(n+1)} \delta_{2j} + u_j^{(n+1)} \delta_{2i})\}, \tag{14b}$$

$$\omega_{ij}^{(n)} = \tfrac{1}{2}\{(u_{i,\alpha}^{(n)} \delta^{\alpha j} - u_{j,\alpha}^{(n)} \delta^{\alpha i}) + (n+1)(u_i^{(n+1)} \delta_{2j} - u_j^{(n+1)} \delta_{2i})\}. \tag{14c}$$

and delta with superscripts is a hybrid isotropic tensor [15].

STRESS DISPLACEMENT RELATIONS

When the material of the plate is anisotropic, and θ the change in temperature, the Duhamel's thermoelastic stress–strain relations generalized for large deformations are given by [16]†

$$\tau_{ij} = c_{ijkl}(\gamma_{kl} - \theta \beta_{kl}) \tag{15}$$

where

$$c_{ijkl} = c_{klij} = c_{jikl}$$

$$\beta_{kl} = \tfrac{1}{2}(\alpha_{kl} + \alpha_{lk}).$$

The coefficients c_{ijkl} are the isothermal elastic coefficients of the anisotropic material and it is assumed that the temperature change is very small. The coefficients α_{kl} characterizing the thermal expansion are tensors of second rank. In the case of isotropic material

† For some critical remarks, see a paper by J. N. Goodier, *Phil. Mag.* **23**, 1017–32 (1937).

we find that

$$\beta_{kl} = \alpha\delta_{kl} = \beta_{lk},$$

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = c_{klij} = c_{jikl},$$

where α is the coefficient of linear thermal expansion. The corresponding stress–strain relations are

$$\tau_{ij} = \lambda(\gamma_{kk} - 3\alpha\theta)\delta_{ij} + 2\mu(\gamma_{ij} - \alpha\theta\delta_{ij}) \tag{16}$$

where λ and μ are Lamé's constants of infinitesimal elasticity and are related to each other through the relation $\lambda/\mu = 2\nu/(1-2\nu)$; $\mu > 0$; $(3\lambda + 2\mu) > 0$; and ν is Poisson's ratio of the material.

Now using the stress–strain relations (15) or (16), expressing γ_{ij} in terms of $\gamma_{ij}^{(n)}$ by using equation (9b), and using the definition of $T_{ij}^{(n)}$, we obtain after integrating with respect to the thickness coordinate x_2, the nth-order stress–strain relations

$$T_{ij}^{(n)} = c_{ijkl} \sum_m B_{mn}(\gamma_{kl}^{(m)} - \theta^{(m)}\beta_{kl}) \tag{17a}$$

for general anisotropic media, and

$$T_{ij}^{(n)} = \sum_m B_{mn}[\lambda(\gamma_{kk}^{(m)} - 3\alpha\theta^{(m)})\delta_{ij} + 2\mu(\gamma_{ij}^{(m)} - \alpha\theta^{(m)}\delta_{ij})] \tag{17b}$$

for isotropic media, where

$$\theta = \sum_m x_2^m \theta^{(m)}(x_1, x_3, t),$$

$$B_{mn} = \frac{(h/2)^{m+n+1}}{(m+n+1)}[1 + \cos(m+n)\pi];\ m, n = 0, 1, 2, \ldots.$$

INTERMEDIATE ORDER EQUATIONS

We will now specialize the general equations to the case when the displacement and temperature field is taken in the form

$$u_i = u_i^{(0)} + x_2 u_i^{(1)}, \tag{18a}$$

$$\theta = \theta^{(0)} + x_2\theta^{(1)}. \tag{18b}$$

Then the only strain components entering in the theory are

$$\gamma_{ij}^{(0)} = e_{ij}^{(0)} + \tfrac{1}{2}\omega_{ki}^{(0)}\omega_{kj}^{(0)} \tag{19a}$$

$$\gamma_{ij}^{(1)} = e_{ij}^{(1)} + \tfrac{1}{2}(\omega_{ki}^{(1)}\omega_{kj}^{(0)} + \omega_{ki}^{(0)}\omega_{kj}^{(1)}) \tag{19b}$$

where from (18a) and (14b, c)

$$\left.\begin{aligned}
&e_{11}^{(0)} = u_{1,1}^{(0)}; \quad e_{22}^{(0)} = u_2^{(1)}; \quad e_{33}^{(0)} = u_{3,3}^{(0)};\\
&2e_{13}^{(0)} = (u_{1,3}^{(0)} + u_{3,1}^{(0)}); \quad 2\omega_{13}^{(0)} = (u_{1,3}^{(0)} - u_{3,1}^{(0)});\\
&2e_{23}^{(0)} = (u_{2,3}^{(0)} + u_3^{(1)}); \quad 2\omega_{23}^{(0)} = (u_{2,3}^{(0)} - u_3^{(1)});\\
&2e_{12}^{(0)} = (u_{2,1}^{(0)} + u_1^{(1)}); \quad 2\omega_{12}^{(0)} = (u_{2,1}^{(0)} - u_1^{(1)});
\end{aligned}\right\} \tag{20}$$

$$e^{(1)}_{11} = u^{(1)}_{1,1}; \qquad e^{(1)}_{22} = 0; \qquad e^{(1)}_{33} = u^{(1)}_{3,3};$$
$$2e^{(1)}_{13} = (u^{(1)}_{1,3}+u^{(1)}_{3,1}); \qquad 2\omega^{(1)}_{13} = (u^{(1)}_{1,3}-u^{(1)}_{3,1}); \tag{21}$$
$$2e^{(1)}_{23} = u^{(1)}_{2,3} = 2\omega^{(1)}_{23}; \qquad 2e^{(1)}_{21} = u^{(1)}_{2,1} = 2\omega^{(1)}_{21}.$$

For frequencies less than the thickness shear mode it is expedient to neglect the thickness stretch displacements $u^{(1)}_2$. However, for free development of Poisson's effect through the thickness, the strain components $\gamma^{(0)}_{22}$ and $\gamma^{(1)}_{2j}$, which are functions of $u^{(1)}_2$ and its partial derivatives, cannot be suppressed. In order that these strains may develop freely, it is necessary that the corresponding stress resultants $T^{(0)}_{22}$ and stress couples $T^{(1)}_{2j}$ must vanish. These conditions can thus be used to eliminate the component $u^{(1)}_2$ without actually suppressing its influence.

From equations (17) and (19), and the definition of B_{mn}, the zero-order stress resultant is given by

$$T^{(0)}_{ij} = hc_{ijkl}(\gamma^{(0)}_{kl} - \theta^{(0)}\beta_{kl}). \tag{22}$$

The condition $T^{(0)}_{22} = 0$ permits the elimination of $(\gamma^{(0)}_{22}-\theta^{(0)}\beta_{22})$ from the zero-order stress resultants, with the result

$$T^{(0)}_{ij} = hg_{ijkl}(\gamma^{(0)}_{kl}-\theta^{(0)}\beta_{kl}) \tag{23a}$$

where $g_{ijkl} = g_{klij} = g_{jikl}$ and in terms of elastic constants c_{ijkl}

$$g_{ijkl} = c_{ijkl}-c_{ij22}c_{22kl} : c_{2222}. \tag{23b}$$

In the case of isotropic medium, equation (23a) reduces to

$$T^{(0)}_{ij} = \frac{hE}{(1-\nu^2)}[\{\nu\gamma^{(0)}_{\alpha\alpha}-\alpha\theta^{(0)}(1+\nu)\}(\delta_{ij}-\delta_{i2}\delta_{j2})+(1-\nu)(\gamma^{(0)}_{ij}-\gamma^{(0)}_{22}\delta_{i2}\delta_{j2})] \tag{23c}$$

where $\gamma^{(0)}_{\alpha\alpha} = \gamma^{(0)}_{11}+\gamma^{(0)}_{33}$.

Similarly, first-order stress couple containing linear variation of temperature $\theta^{(1)}$ is given by

$$T^{(1)}_{ij} = \frac{h^3}{12}c_{ijkl}(\gamma^{(1)}_{kl}-\theta^{(1)}\beta_{kl}) \tag{24}$$

where the first-order strain components are as in (19b). In order to allow free development of thickness strains $\gamma^{(1)}_{2j}$, we now set $T^{(1)}_{2j} = 0$, and use these three equations to determine $\gamma^{(1)}_{2j}$, which are then used in equation (24) to obtain the stress–strain relations between the remaining components $T^{(1)}_{\alpha\beta}$ and $\gamma^{(1)}_{\alpha\beta}$. This process of elimination thus finally leads to the result:

$$T^{(1)}_{ij} = \frac{h^3}{12}\Pi_{ijkl}(\gamma^{(1)}_{kl} - \theta^{(1)}\beta_{kl}) \tag{25a}$$

where

$$\Pi_{ijkl} = d_{ijkl}-d_{ij12}d_{12kl} : d_{1212},$$
$$d_{ijkl} = g_{ijkl}-g_{ij23}g_{23kl} : g_{2323}, \tag{25b}$$
$$g_{ijkl} = c_{ijkl}-c_{ij22}c_{22kl} : c_{2222}.$$

From equation (25b) we note that $\Pi_{ijkl} = \Pi_{klij} = \Pi_{jikl}$ and $\Pi_{ij22} = \Pi_{ij23} = \Pi_{ij21} = 0$. Therefore, $T_{2j}^{(1)}$ are zero and $\gamma_{2j}^{(1)}$ are eliminated from the remaining three equations of stress-couples. Effectively we are, therefore, left with the relation

$$T_{\alpha\beta}^{(1)} = \frac{h^3}{12}\Pi_{\alpha\beta\sigma\rho}(\gamma_{\sigma\rho}^{(1)} - \theta^{(1)}\beta_{\sigma\rho}) \tag{26}$$

where $\Pi_{\alpha\beta\sigma\rho}$ are obtained from Π_{ijkl} by replacing the subscripts $ijkl$ by $\alpha\beta\sigma\rho$.

The elimination process is not yet complete because $u_{2,1}^{(1)}$ and $u_{2,3}^{(1)}$ enter in $\gamma_{\sigma\rho}^{(1)}$ through $\omega_{2\rho}^{(1)}$. But we note from (21) that $\omega_{2\rho}^{(1)} = e_{2\rho}^{(1)}$. According to our original hypothesis terms of the type $e_{2\rho}^{(1)}\omega_{2\sigma}^{(0)}$, $\omega_{2\rho}^{(0)}e_{2\sigma}^{(1)}$ can be neglected in comparison with products of rotation, and therefore from (19b), we can write

$$\gamma_{\sigma\rho}^{(1)} = e_{\sigma\rho}^{(1)} + \tfrac{1}{2}(\omega_{\gamma\sigma}^{(1)}\omega_{\gamma\rho}^{(0)} + \omega_{\gamma\rho}^{(1)}\omega_{\gamma\sigma}^{(0)}) \tag{27a}$$

which is equivalent to

$$\gamma_{\sigma\rho}^{(1)} = e_{\sigma\rho}^{(1)} + \omega_{31}^{(1)}\omega_{31}^{(0)}\delta_{\sigma\rho}. \tag{27b}$$

The first-order stress-couples in terms of strain-components therefore take the form:

$$T_{\alpha\beta}^{(1)} = \frac{h^3}{12}\Pi_{\alpha\beta\sigma\rho}(\gamma_{\sigma\rho}^{(1)} - \theta^{(1)}\beta_{\sigma\rho}), \tag{28a}$$

$$\gamma_{\sigma\rho}^{(1)} = e_{\sigma\rho}^{(1)} + \omega_{31}^{(1)}\omega_{31}^{(0)}\delta_{\sigma\rho}, \tag{28b}$$

$$\Pi_{\alpha\beta\sigma\rho} = d_{\alpha\beta\sigma\rho} - d_{\alpha\beta12}d_{12\sigma\rho} : d_{1212}, \tag{28c}$$

$$d_{\alpha\beta\sigma\rho} = g_{\alpha\beta\sigma\rho} - g_{\alpha\beta23}g_{23\sigma\rho} : g_{2323}, \tag{28d}$$

$$g_{\alpha\beta\sigma\rho} = c_{\alpha\beta\sigma\rho} - c_{\alpha\beta22}c_{22\sigma\rho} : g_{2222}. \tag{28e}$$

For the case of isotropic medium, $\Pi_{\alpha\beta\sigma\rho} = g_{\alpha\beta\sigma\rho}$ (isotropic) and using the property of fourth rank isotropic tensors, we find

$$T_{\alpha\beta}^{(1)} = \frac{h^3E}{12(1-\nu^2)}[\{\nu e_{\rho\rho}^{(1)} + (1+\nu)(\omega_{31}^{(1)}\omega_{31}^{(0)} - \alpha\theta^{(1)})\}\delta_{\alpha\beta} + (1-\nu)e_{\alpha\beta}^{(1)}] \tag{28f}$$

where $e_{\rho\rho}^{(1)}$ is the areal dilation of first order. In the case of orthotropy $\Pi_{\alpha\beta\sigma\rho} = g_{\alpha\beta\sigma\rho}$.

The incorrect distribution of displacements assumed in the series expansion (18), affect the frequencies mainly through the thickness-shear strains $(\gamma_{2\alpha}^{(0)} - \theta^{(0)}\beta_{2\alpha})$. In order to adjust the thickness-shear frequencies to their correct value, and also to compensate for the other higher-order terms which have been neglected in the theory, the thickness-shear strains $(\gamma_{21}^{(0)} - \theta^{(0)}\beta_{21})$ and $(\gamma_{23}^{(0)} - \theta^{(0)}\beta_{23})$ are replaced by $k_{(1)}(\gamma_{21}^{(0)} - \theta^{(0)}\beta_{21})$ and $k_{(3)}(\gamma_{23}^{(0)} - \theta^{(0)}\beta_{23})$ respectively. Thus writing $(\gamma_{ij}^{(0)} - \theta^{(0)}\beta_{ij}) = k_{(i+j-2)}^{m}(\gamma_{ij}^{(0)} - \theta^{(0)}\beta_{ij})$ in the zero-order strain-energy density of the plate, and noting that $T_{ij}^{(0)} = \frac{1}{2}(\partial W^{(0)}/\partial\gamma_{ij}^{(0)} + \partial W^{(0)}/\partial\gamma_{ji}^{(0)})$, we find that when shear correction is taken into account, the zero-order resultants take the form [10]:

$$T_{ij}^{(0)} = hg_{ijkl}^{*}(\gamma_{kl}^{(0)} - \theta^{(0)}\beta_{kl}), \tag{29}$$

where

$$g_{ijkl}^{*} = k_{(i+j-2)}^{m}g_{ijkl}k_{(k+l-2)}^{n},$$

$$m = \cos^2(ij\pi/2);\ n = \cos^2(kl\pi/2),$$

and no summation is implied over parenthetic subscripts.

In the theory developed so far we have assumed that elongations and shear are negligible compared to unity, and we have imposed no restriction on the magnitude of the rotations. When the rotations of the elements of the plate are also small, compared to unity, further simplification can be achieved in the strain–displacement relations.† It has been shown by Novozhilov that in this case the derivatives $u^{(0)}_{\alpha,\alpha}$ and $u^{(0)}_{\alpha,\beta}$ can be regarded as quantities of the same order of magnitude as $\gamma^{(0)}_{\alpha\alpha}$ and $\gamma^{(0)}_{\alpha\beta}$ [13]. This then permits us to drop all non-linear terms involving derivatives of $u^{(0)}_{\alpha}$, in the strain components $\gamma^{(0)}_{ij}$ and $\gamma^{(1)}_{\alpha\beta}$ defined by equations (19a) and (28b). With this additional approximation, the strain displacement relations take the form:

$$\gamma^{(0)}_{\alpha\beta} = e^{(0)}_{\alpha\beta}+\tfrac{1}{2}\omega^{(0)}_{2\alpha}\omega^{(0)}_{2\beta}, \tag{30a}$$

$$\gamma^{(0)}_{2\alpha} = e^{(0)}_{2\alpha} = \tfrac{1}{2}(u^{(0)}_{2,\alpha}+u^{(1)}_{\alpha}), \tag{30b}$$

$$\gamma^{(1)}_{\alpha\beta} = e^{(1)}_{\alpha\beta} = \tfrac{1}{2}(u^{(1)}_{\alpha,\beta}+u^{(1)}_{\beta,\alpha}), \tag{30c}$$

and

$$e^{(0)}_{\alpha\beta} = \tfrac{1}{2}(u^{(0)}_{\alpha,\beta}+u^{(0)}_{\beta,\alpha}), \qquad \omega^{(0)}_{2\alpha} = \tfrac{1}{2}(u^{(0)}_{2,\alpha}-u^{(1)}_{\alpha}).$$

Now it is not difficult to verify that the six components $e^{(n)}_{\alpha\beta}$ for $n = 0,1$ and the two components $e^{(0)}_{2\alpha}$, satisfy the relations

$$\tfrac{1}{4}\delta^{2\sigma\alpha}_{2\beta\rho}\{e^{(n)}_{\alpha\beta,\sigma\rho}+e^{(n)}_{\sigma\rho,\alpha\beta}-e^{(n)}_{\sigma\beta,\alpha\rho}-e^{(n)}_{\alpha\rho,\sigma\beta}\} = 0, \tag{31a}$$

$$\tfrac{1}{4}\delta^{2i\alpha}_{\beta j\rho}\{e^{(1)}_{\alpha\beta,\rho}+e^{(0)}_{2\rho,\alpha\beta}-e^{(0)}_{2\beta,\alpha\rho}-e^{(1)}_{\alpha\rho,\beta}\} = 0, \tag{31b}$$

where $\delta^{\cdots}_{\cdots}$ is the generalized Kronecker delta, completely asymmetric in subscripts and superscripts. Using equation (30a) in (31a), it can then be shown that the strain components $\gamma^{(0)}_{\alpha\beta}$ satisfy the relation

$$\delta^{2\alpha\sigma}_{2\rho\beta}\{\gamma^{(0)}_{\alpha\beta,\sigma\rho}+\gamma^{(0)}_{\sigma\rho,\alpha\beta}-\gamma^{(0)}_{\alpha\rho,\sigma\beta}-\gamma^{(0)}_{\sigma\beta,\alpha\rho}\} = 2\delta^{\alpha\sigma}_{\rho\beta}(\omega^{(0)}_{2\alpha}\omega^{(0)}_{2\beta})_{,\sigma\rho}. \tag{32a}$$

In the particular case when shear deformation is neglected, $u^{(1)}_{\alpha} = -u^{(0)}_{2,\alpha}$, and the equation takes the form:

$$\delta^{\alpha\sigma}_{\rho\beta}\{\gamma^{(0)}_{\alpha\beta,\sigma\rho}+\gamma^{(0)}_{\sigma\rho,\alpha\beta}-\gamma^{(0)}_{\alpha\rho,\sigma\beta}-\gamma^{(0)}_{\sigma\beta,\alpha\rho}\} = 2\delta^{\alpha\sigma}_{\beta\rho}u^{(0)}_{2,\alpha\beta}u^{(0)}_{2,\sigma\rho} \tag{32b}$$

where $\delta^{\alpha\sigma}_{\beta\rho} = (\delta_{\alpha\beta}\delta_{\sigma\rho}-\delta_{\alpha\rho}\delta_{\sigma\beta})$.

STRESS AND DISPLACEMENT EQUATIONS

When $T^{(0)}_{22} = 0$; $T^{(n)}_{2j} = 0$ for $n > 0$; $T^{(n)}_{ij} = 0$ for $n > 1$; $\dot{u}^{(1)}_{2} = 0$; $u^{(n)}_{j} = 0$ for $n > 1$; $\omega^{(n)}_{2\alpha} \approx 0$ for $n > 0$; $\omega^{(n)}_{\alpha\beta} \ll 1$ for $n \geq 0$; and the only non-zero mean rotations are $\omega^{(0)}_{2\alpha}$, the stress equations of motion (12) and (13) take the form:

$$\{T^{(0)}_{\alpha\beta}+\omega^{(0)}_{\alpha 2}T^{(0)}_{2\beta}\}_{,\beta}+G^{(0)}_{2\alpha}+\rho X^{(0)}_{\alpha} = \rho h\ddot{u}^{(0)}_{\alpha} \tag{33a}$$

$$\{T^{(0)}_{2\beta}+\omega^{(0)}_{2\alpha}T^{(0)}_{\alpha\beta}\}_{,\beta}+G^{(0)}_{22}+\rho X^{(0)}_{2} = \rho h\ddot{u}^{(0)}_{2}, \tag{33b}$$

† Smallness of strains and rotations in comparison with unity does not imply that they are of the same order of magnitude.

$$T^{(1)}_{\beta\alpha,\alpha} - T^{(0)}_{2\beta} + G^{(1)}_{2\beta} + \rho X^{(1)}_{\beta} = \rho \frac{h^3}{12} \ddot{u}^{(1)}_{\beta}. \tag{33c}$$

In these equations we have neglected the projected components on the plane of the plate of the surface loads $G^{(0)}_{2i}$ and surface moments $G^{(1)}_{22}$. The surface loads and the surface moments are given by the relations

$$G^{(n)}_{i2} = (h/2)^n\{\tau_{i2}]_{+h/2} - (-1)^n \tau_{i2}]_{-h/2}\}.$$

From these equations the body forces like $X^{(0)}_{\alpha}$ and $X^{(1)}_{\beta}$ can be neglected and equation (32b) can be combined with equation (36c) with the result:

$$\{T^{(0)}_{\alpha\beta} + \omega^{(0)}_{\alpha 2} T^{(0)}_{2\beta}\}_{,\beta} + G^{(0)}_{2\alpha} = \rho h \ddot{u}^{(0)}_{\alpha}, \tag{34a}$$

$$\{T^{(1)}_{\beta\alpha,\alpha} + \omega^{(0)}_{2\alpha} T^{(0)}_{\alpha\beta}\}_{,\beta} + G^{(0)}_{22} + G^{(1)}_{2\beta,\beta} + \rho X^{(0)}_{2} = \rho h\left(\ddot{u}^{(0)}_{2} + \frac{h^2}{12}\ddot{u}^{(1)}_{\beta,\beta}\right). \tag{34b}$$

The first set of two equations (34a) describe the extensional motion of the plate in its own plane, and the third equation (34b) describes the flexural motion with rotatory inertia included.

For anisotropic medium

$$T^{(0)}_{2\beta} = hg^{*}_{2\beta\sigma\rho}(\gamma^{(0)}_{\sigma\rho} - \theta^{(0)}\beta_{\sigma\rho}) + 2hg^{*}_{2\beta 2\sigma}(e^{(0)}_{2\sigma} - \theta^{(0)}\beta_{2\sigma}), \tag{35a}$$

$$T^{(0)}_{\alpha\beta} = hg^{*}_{\alpha\beta\sigma\rho}(\gamma^{(0)}_{\sigma\rho} - \theta^{(0)}\beta_{\sigma\rho}) + 2hg^{*}_{\alpha\beta 2\sigma}(e^{(0)}_{2\sigma} - \theta^{(0)}\beta_{2\sigma}), \tag{35b}$$

$$T^{(1)}_{\alpha\beta} = \frac{h^3}{12}\Pi_{\alpha\beta\sigma\rho}(e^{(1)}_{\sigma\rho} - \theta^{(1)}\beta_{\sigma\rho}), \tag{35c}$$

and for isotropic medium these relations reduce to

$$T^{(0)}_{2\beta} = 2\mu h k^2_{(\beta)} e^{(0)}_{2\beta}, \tag{36a}$$

$$T^{(0)}_{\alpha\beta} = 2\mu h(1-\nu)^{-1}[\{\nu\gamma^{(0)}_{\rho\rho} - \alpha(1+\nu)\theta^{(0)}\}\delta_{\alpha\beta} + (1-\nu)\gamma^{(0)}_{\alpha\beta}], \tag{36b}$$

$$T^{(1)}_{\alpha\beta} = \tfrac{1}{6}\mu h^3(1-\nu)^{-1}[\{\nu e^{(1)}_{\rho\rho} - \alpha(1+\nu)\theta^{(1)}\}\delta_{\alpha\beta} + (1-\nu)e^{(1)}_{\alpha\beta}]. \tag{36c}$$

In the case of orthotropic medium $\Pi_{\alpha\beta\sigma\rho} = g^{*}_{\alpha\beta\sigma\rho} = g_{\alpha\beta\sigma\rho}$ and $g^{*}_{\alpha\beta 2\sigma}$ is zero and the corresponding stress–strain relations can be easily written from the set of relations (35). The strain tensors entering in these relations is defined in terms of displacements in (30).

In the case of low-frequency flexural vibrations, it is generally sufficient to neglect extensional inertia $\ddot{u}^{(0)}_{\alpha}$ and linearize the equation by dropping the term $\omega^{(0)}_{\alpha 2} T^{(0)}_{2\beta}$ in comparison with $T^{(0)}_{\alpha\beta}$. In the absence of body force $X^{(0)}_{\alpha}$ and surface loads $G^{(0)}_{2\alpha}$, it is easy to introduce a stress function $T^{(0)}_{\alpha\beta} = \delta^{\alpha\gamma}_{\beta\sigma}\phi_{,\sigma\rho}$ which satisfies the equation $T^{(0)}_{\alpha\beta,\beta} = 0$ identically. In terms of stress function ϕ and displacements $u^{(0)}_{2}$; $u^{(1)}_{\alpha}$, the remaining three equations (33 b, c), in the case of orthotropy take the form:

$$\tfrac{1}{2}\phi_{,\sigma\rho}\delta^{\rho\alpha}_{\sigma\beta}(u^{(0)}_{2,\alpha} - u^{(1)}_{\alpha})_{,\beta} + hg^{*}_{2\beta 2\alpha}\psi_{\alpha,\beta} + G^{(0)}_{22} + \rho X^{(0)}_{2} = \rho h \ddot{u}^{(0)}_{2}, \tag{37a}$$

$$\frac{h^3}{12}g_{\alpha\beta\sigma\rho}(u^{(1)}_{\sigma,\rho\alpha} - \beta_{\sigma\rho}\theta^{(1)}_{,\alpha}) - hg^{*}_{2\beta 2\alpha}\psi_{\alpha} + G^{(1)}_{2\beta} + \rho X^{(1)}_{\beta} = \rho\frac{h^3}{12}\ddot{u}^{(1)}_{\beta}, \tag{37b}$$

where $\psi_{\alpha} = (u^{(0)}_{2,\alpha} + u^{(1)}_{\alpha} - 2\beta_{2\alpha}\theta^{(0)})$, and for orthotropy $g^{*}_{2\beta 2\alpha} = k_{(\beta)}c_{2\beta\alpha 2}k_{(\alpha)}$. These three equations contain four unknowns and to obtain the fourth equation we use the identity

(32a) by expressing the strain components $\gamma^{(0)}_{\alpha\beta}$ in terms of the stress function ϕ. For an orthotropic medium, the scalar invariant ϕ and the strain components $\gamma^{(0)}_{\alpha\beta}$ are related to each other through the relation

$$h\gamma^{(0)}_{\alpha\beta} = h\theta^{(0)}\beta_{\alpha\beta} + C_{\alpha\beta\sigma\rho}\delta^{\sigma\lambda}_{\rho\kappa}\phi_{,\lambda\kappa} \tag{38}$$

where $C_{\alpha\beta\sigma\rho} = G^{\alpha\beta\sigma\rho}: |g_{\alpha\beta\sigma\rho}|$; and $G^{\alpha\beta\sigma\rho}$ is the cofactor of $g_{\sigma\rho\alpha\beta}$ in the determinant $g_{\alpha\beta\sigma\rho}$, so that for inner product of symmetric tensors, $g_{\alpha\beta\sigma\rho}C_{\sigma\rho\lambda\mu} = \frac{1}{2}(\delta_{\alpha\lambda}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\lambda}) = \delta_{\alpha\lambda}\delta_{\beta\mu}$. Now introducing (38) in the left hand side of equation (32a), it can be shown through proper permutation of the dummy indices that

$$\frac{h}{4}\delta^{2\alpha\sigma}_{2\rho\beta}\{\gamma^{(0)}_{\alpha\beta,\sigma\rho}+\gamma^{(0)}_{\sigma\rho,\alpha\beta}-\gamma^{(0)}_{\alpha\rho,\sigma\beta}-\gamma^{(0)}_{\sigma\beta,\alpha\rho}\} = h\delta^{\alpha\sigma}_{\rho\beta}\theta^{(0)}_{,\sigma\rho}\beta_{\alpha\beta}+C_{\alpha\beta\kappa\nu}\delta^{\alpha\sigma}_{\rho\beta}\delta^{\kappa\lambda}_{\nu\mu}\phi_{,\sigma\rho\lambda\mu}. \tag{39}$$

Then the remaining equation which together with equations (37a, b) constitutes the complete system of four simultaneous equations in four unknowns, is given by

$$\delta^{\alpha\rho}_{\sigma\beta}\{C_{\alpha\beta\kappa\nu}\delta^{\kappa\lambda}_{\nu\mu}\phi_{,\sigma\rho\lambda\mu}+h\beta_{\alpha\beta}\theta^{(0)}_{,\sigma\rho}-\frac{h}{8}[(u^{(0)}_{2,\alpha}-u^{(1)}_{\alpha})(u^{(0)}_{2,\beta}-u^{(1)}_{\beta})]_{,\sigma\rho}\} = 0 \tag{37c}$$

where

$$C_{\alpha\beta\kappa\nu}\delta^{\kappa\lambda}_{\nu\mu}\phi_{,\sigma\rho\lambda\mu} = C_{\alpha\beta 11}\phi_{,33\sigma\rho}-2C_{\alpha\beta 13}\phi_{,13\sigma\rho}+C_{\alpha\beta 33}\phi_{,11\sigma\rho}.$$

These four equations for large flexural motion of thin orthotropic plates can be easily reduced to the case when the plate is isotropic. For it can be shown that in the case of isotropy the various constants entering in the system of equations (37) are:

$$\begin{aligned}
g^{*}_{2\beta\alpha 2}\beta_{2\alpha} &= 0,\\
g_{\alpha\beta\sigma\rho}\beta_{\sigma\rho} &= 2\alpha(\lambda'+\mu)\delta_{\alpha\beta},\\
|g_{\alpha\beta\sigma\rho}| &= 16\mu^2E(\lambda+\mu)/(\lambda+2\mu),\\
G^{1111} &= G^{1133}+2G^{1313} = G^{3333} = 16\mu^2(\lambda+\mu)/(\lambda+2\mu),\\
C_{1111} &= C_{1133}+2C_{1313} = C_{3333} = 1/E,\\
C_{1113} &= C_{1333} = 0;\; G^{1113} = G^{1333} = 0,\\
g_{\alpha\beta\sigma\rho} &= \lambda'\delta_{\alpha\beta}\delta_{\sigma\rho}+\mu(\delta_{\alpha\sigma}\delta_{\beta\rho}+\delta_{\alpha\rho}\delta_{\beta\sigma}),\\
g^{*}_{2\beta\sigma 2} &= \mu k_{(\beta)}\delta_{\beta\sigma}k_{(\sigma)},\\
\lambda' &= 2\mu\lambda/(\lambda+2\mu);\; (\lambda'+\mu) = \mu(1+\nu)/(1-\nu);\; E = 2\mu(1+\nu).
\end{aligned}$$

The four simultaneous equations, for the case of isotropy, containing both shear correction and rotatory inertia, then reduce to the form:

$$\mu'h(u^{(0)}_{2,\beta}+u^{(1)}_{\beta})_{,\beta}+\tfrac{1}{2}\phi_{,\sigma\rho}\delta^{\rho\alpha}_{\sigma\beta}(u^{(0)}_{2,\alpha}-u^{(1)}_{\alpha})_{,\beta}+G^{(0)}_{22}+\rho X^{(0)}_{2} = \rho h\ddot{u}^{(0)}_{2}, \tag{40a}$$

$$\frac{h^3}{12}(\mu u^{(1)}_{\beta,\alpha}+\lambda''u^{(1)}_{\alpha,\beta})_{,\alpha}-\mu'h(u^{(0)}_{2,\beta}+u^{(1)}_{\beta})-\tfrac{1}{6}\alpha h^3\lambda''\theta^{(1)}_{,\beta}+G^{(1)}_{2\beta}+\rho X^{(1)}_{\beta} = \rho\frac{h^3}{12}\ddot{u}^{(1)}_{\beta}, \tag{40b}$$

$$\phi_{,\alpha\beta\alpha\beta}+\alpha hE\theta^{(0)}_{,\alpha\alpha}-\tfrac{1}{8}hE\delta^{\alpha\rho}_{\beta\sigma}\{(u^{(0)}_{2,\alpha}-u^{(1)}_{\alpha})(u^{(0)}_{2,\beta}-u^{(1)}_{\beta})\}_{,\sigma\rho} = 0, \tag{40c}$$

where $\lambda'' = \lambda'+\mu$; $\mu' = \mu k^2_{(\beta)}$.

At extremely low frequencies, it is permissible to neglect shear deformation from the equations of motion. This then permits us to write

$$u_\alpha^{(1)} = -u_{2,\alpha}^{(0)} \equiv -w_{,\alpha}$$

where we have in accordance with the standard notation replaced the transverse displacement $u_2^{(0)}$ by w. Then in the case of orthotropy, when shear deformation is neglected, the large deflection equations of motion, in the presence of thermal field, take the form

$$\frac{h^3}{12}g_{\alpha\beta\sigma\rho}(w_{,\sigma\rho}+\beta_{\sigma\rho}\theta^{(1)})_{,\alpha\beta}-w_{,\alpha\beta}\delta_{\beta\sigma}^{\alpha\rho}\phi_{,\sigma\rho}+\rho h\left(\ddot{w}-\frac{h^2}{12}\ddot{w}_{,\beta\beta}\right) = F_2, \tag{41a}$$

$$\delta_{\sigma\beta}^{\alpha\rho}[C_{\alpha\beta\kappa\nu}\delta_{\nu\mu}^{\kappa\lambda}\phi_{,\sigma\rho\lambda\mu}+h\beta_{\alpha\beta}\theta_{,\sigma\rho}^{(0)}+\tfrac{1}{2}hw_{,\alpha\beta}w_{,\sigma\rho}] = 0, \tag{41b}$$

where $F_2 = G_{22}^{(0)}+G_{22,\beta}^{(1)}+\rho(X_2^{(0)}+X_{\beta,\beta}^{(1)})$. In the case of isotropy,

$$D(w_{,\beta\beta}+\alpha(1+\nu)\theta^{(1)})_{,\alpha\alpha}-w_{,\alpha\beta}\delta_{\beta\sigma}^{\alpha\rho}\phi_{,\sigma\rho}+\rho h\left(\ddot{w}-\frac{h^2}{12}\ddot{w}_{,\beta\beta}\right) = F_2, \tag{42a}$$

$$\phi_{,\alpha\beta\alpha\beta}+\alpha hE\theta_{,\alpha\alpha}^{(0)}-\tfrac{1}{2}hE\delta_{\sigma\beta}^{\alpha\rho}w_{,\alpha\beta}w_{,\sigma\rho} = 0, \tag{42b}$$

where the flexural rigidity $D = Eh^3/12(1-\nu^2)$. The solution of these non-linear equations of motion require that these be solved simultaneously for w and ϕ. Once w and ϕ are determined, the extensional displacements can be determined from the relations

$$he_{\alpha\beta}^{(0)} = h\theta^{(0)}\beta_{\alpha\beta}+C_{\alpha\beta\sigma\rho}\delta_{\rho\mu}^{\sigma\lambda}\phi_{,\lambda\mu}-\tfrac{1}{2}hw_{,\alpha}w_{,\beta} \tag{43a}$$

in the case of orthotropy, or

$$he_{\alpha\beta}^{(0)} = \alpha h\theta^{(0)}\delta_{\alpha\beta}+\frac{1}{2\mu}\left[\delta_{\beta\sigma}^{\alpha\rho}\phi_{,\sigma\rho}-\frac{\nu}{1+\nu}\phi_{,\rho\rho}\delta_{\alpha\beta}\right]-\tfrac{1}{2}hw_{,\alpha}w_{,\beta} \tag{43b}$$

in the case of isotropy. Further, once ϕ is known, the planar stress field can be easily determined in each case from the relation $T_{\alpha\beta}^{(0)} = \delta_{\beta\rho}^{\alpha\sigma}\phi_{,\sigma\rho}$.

The equations of static equilibrium can be easily obtained from these equations of motion by dropping the inertia terms. Then equation (37) yields the equation of equilibrium of finite deflection of orthotropic plates and equation (40) gives the equation of equilibrium of isotropic plates; both containing shear correction. If on the other hand, shear correction be neglected, and also inertia terms dropped, then equation (41) yields the equation of equilibrium for large deflection of orthotropic thin plates, and equation (42) gives the corresponding equation for isotropic plates. It may be noticed that the effect of neglecting the shear deformation results in considerable simplicity and requires only the solution of two simultaneous equations instead of four as in equation (37) on equation (40). It is not difficult to recognize, that if we also drop thermal distribution from equation (42), then we are led to Föppl–Kármán equations of equilibrium of finite deflection of large thin isotropic plates, [5, 6, 14]. Under similar conditions, our equation (41) is a generalization of Föppl–Kármán equation for orthotropic plates. Both these equations are special cases of general equations (37) and (40), for orthotropy or isotropy, containing shear correction and allowing for a thermal field, rotatory inertia and flexural motion. The correction factors $k_{(\beta)}$ have the same meaning as those used by Mindlin [10].

Concerning the system of equations (37), it can be seen that we require initial values of $u_2^{(0)}$, $u_\alpha^{(1)}$, ϕ and of $\dot{u}_2^{(0)}$, $\dot{u}_\alpha^{(1)}$, $\dot{\phi}$. In addition to the initial values, we find from the second of equations (12) and (13), that at every point on the two surfaces of the plate, either surface forces or surface displacements be specified from each of the following three pairs

$$G_{2\alpha}^{(0)} \quad \text{or } u_\alpha^{(0)}; \qquad G_{22}^{(0)} \quad \text{or } u_2^{(0)}; \qquad G_{2\alpha}^{(1)} \quad \text{or } u_\alpha^{(1)}.$$

Furthermore, the third of equations (12) and (13) show that the system of partial differential equations to be solved is subject to the following edge conditions

$$f_\alpha^{(0)}(s) = \nu_\beta T_{\alpha\beta}^{(0)} \qquad \text{or } u_\alpha^{(0)} = U_\alpha^{(0)}(s),$$
$$f_2^{(0)}(s) = \nu_\beta (T_{2\beta}^{(0)} + \omega_{2\alpha}^{(0)} T_{\alpha\beta}^{(0)}) \qquad \text{or } u_2^{(0)} = U_2^{(0)}(s),$$
$$f_\alpha^{(1)}(s) = \nu_\beta T_{\alpha\beta}^{(1)} \qquad \text{or } u_\alpha^{(1)} = U_\alpha^{(1)}(s),$$

where the displacements $U_i^{(0)}(s)$, $U_\alpha^{(1)}(s)$ and the components of the applied forces $f_i^{(0)}(s)\,\mathrm{d}s$ and applied moments $f_\alpha^{(1)}(s)\,\mathrm{d}s$, are known functions of arc parameter s of the bounding curve C.

Writing $\nu_\beta = \varepsilon_{\beta\gamma}\,\mathrm{d}x_\gamma/\mathrm{d}s$; $T_{\alpha\beta}^{(0)} = \delta_{\beta\rho}^{\alpha\sigma}\phi_{,\sigma\rho}$ it follows that

$$\varepsilon_{\alpha\beta}\frac{\mathrm{d}}{\mathrm{d}s}\phi_{,\beta} = f_\alpha^{(0)}(s) \qquad \text{or } \phi_{,\beta} = \varepsilon_{\alpha\beta}\int_{s_0}^{s} f_\alpha^{(0)}(s)\,\mathrm{d}s \equiv g(s).$$

Thus in terms of stress function, $\phi_{,\beta}$ must be a known function of s. In fact knowledge of $\phi_{,\beta}(s)$ permits one to determine the values of $\phi(s)$ and its normal derivative on C. It is understood throughout that when the region is multiply connected, the boundaries C consist of $(n+1)$ simple and smooth curves C_i with $C = C_{n+1} + C_1 + \ldots + C_n$.

Acknowledgements—The material presented in this paper is based on the original contributions of Professor R. D. Mindlin whose active interest in this work and useful suggestions are gratefully acknowledged. The author also wishes to thank Mrs. G. C. Kneen for the careful preparation of the manuscript.

REFERENCES

[1] A. NÁDAI, *Elastische Platten*, pp. 264–68. Springer, Berlin (1925).
[2] W. VOIGT, *Lehrbuch der Krystallphysik*, pp. 675–98, 763–91. B. G. Teubner, Leipzig und Berlin (1910).
[3] A. L. CAUCHY, *Exercices de Mathematique* **4**, 1 (1839).
[4] R. D. MINDLIN, *J. appl. Phys.* **22**, 316 (1951).
[5] A. FÖPPL, *Vorlesungen über technische Mechanik*, Bd. 5, Leipzig §24 (1907).
[6] T. VON KÁRMÁN, *Ency. d. Math. Wiss.*, Bd. IV. 2, II, Leipzig §8 (1910).
[7] S. TIMOSHENKO, *Theory of Plates and Shells*, pp. 301, 342. McGraw-Hill (1940).
[8] S. TIMOSHENKO, *Phil. Mag.*, Ser. 6, **41**, 744 (1921).
[9] R. D. MINDLIN, *J. appl. Mech.* **18**, 31 (1951).
[10] R. D. MINDLIN, *Quart. appl. Math.* **19**, 51 (1961).
[11] G. KIRCHHOFF, *Crelle J.* **40**, 51 (1850).
[12] R. D. MINDLIN, *Proc. 1st Symposium Naval Structural Mechanics*, pp. 199–232. Pergamon Press, New York (1960); see also R. D. MINDLIN and D. GAZIS, *4th U.S. Nat. Congress Appl. Mech.*, Berkeley, Calif., June, 1962.
[13] V. V. NOVOZHILOV, *Foundations of the Nonlinear Theory of Elasticity*. Graylock Press (1953).
[14] A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, fourth edition, pp. 166, 557. Dover Publications, New York (1944).
[15] A. G. MCCONNEL, *Applications of Tensor Analysis*, pp. 193–197. Dover Publications, New York (1947).
[16] J. M. C. DUHAMEL, *Memoires par divers Savants, Paris*, **5**, 440 (1838); see also F. NEUMANN, *Abhandlungen der deutschen Akademie der Wissenschaften, Berlin*, Part 2, p. 254 (1841).

(*Received* 11 *January* 1965)

Zusammenfassung—Mit Hilfe der nichtlinearen, dreidimensionalen Schwingungstheorie für die endliche Elastizität wird für anisotrope, dünne Platten ein zweidimensionales Wärme—und Bewegungsgleichungs-system abgeleitet. Dabei verwendet man die variationsrechnerische Kirchhoff'sche Methode und eine andere Methode, welche sich auf die in Dickekoordinaten ausgedrückte Entwicklung der Verschiebefunktionen gründet. Die Nichtlinearität wird auf der Grundlage geometrischer Erwägungen ausgedrückt, indem man bei der Definition der Spannungstensoren die Drehwinkel in Betracht zieht, während man die physikalische Linearität mit Hilfe des auf grosse Deformationen verallgemeinerten Duhamel'schen Gestzes der anisotropen Thermoeleastizität ausdrückt.

Абстракт—Система двухмерных термических уравнений движения для неизотропных тонких пластин, получается из нелинейной трехмерной колебательной теории конечной упругости, применением методов основанных на разложении функций смещения в условиях координат толщины, а также вариационного метода Кирхгоффа.

Нелинейность, основанная на геометрических соображениях, выражается тем, что принимаются во внимание углы вращения в определении тензора деформации, в то время как физическая линейность выражается применением закона Дюгамеля об неизотропной термоэластичности обобщенной для больших деформаций.